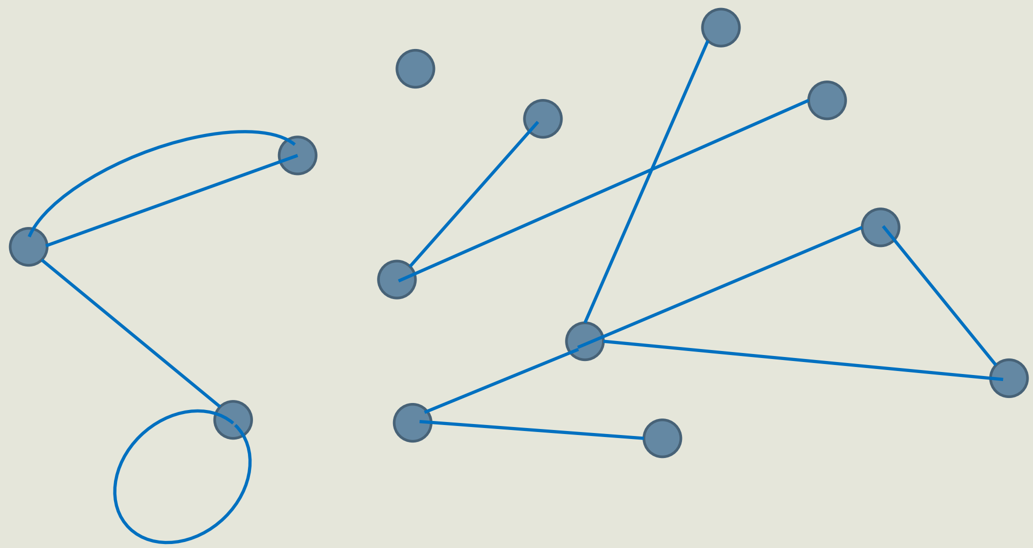
Graphs

Lecture 7   Feb 21, 2021

Definition

- A **graph** G is a made of a finite collection of vertices or nodes V and collection E of edges between the nodes of V

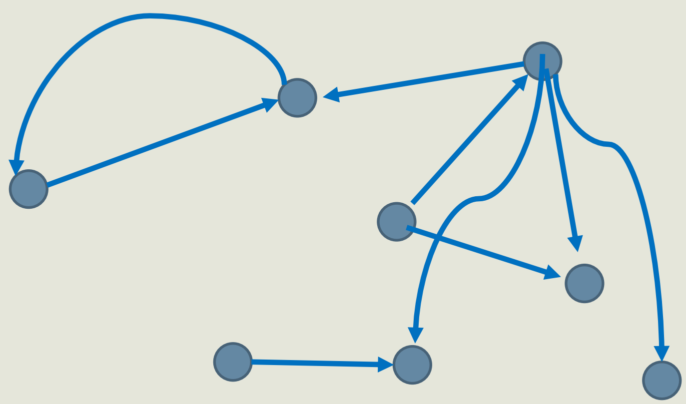
Schematic description : (It does not matter how you draw the edges: straight or curly)



Definition

- An <u>oriented</u> graph G is a made of a finite collection of <u>vertices or nodes</u> V and collection E of <u>oriented</u> edges between the nodes of V

Schematic description :

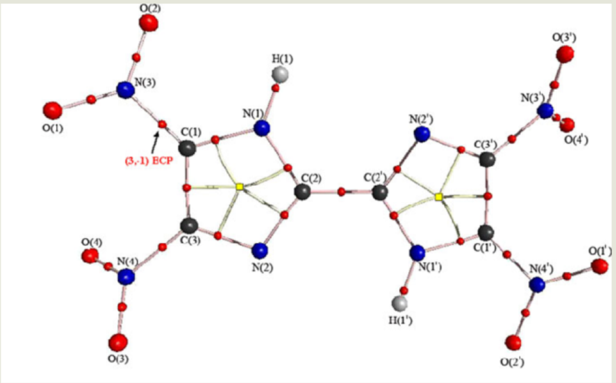


In Applications:

- Facebook: People are nodes; Friendships are edges

- Facebook is a regular graph (not oriented): if I am friend with you, you are also friend with me

- Instagram is an oriented graph: People are nodes; If A follows B, there is an oriented edge (relation) from person A to B
- Navigation maps can be seen as graphs: cities as nodes, roads as edges
- Chemistry: Graph of Molecules



(Picture: Courtesy of https://www.researchgate.net/figure/The-molecular-graph-showing-the-3-1-and-3-1-critical-points-of-the-TNBI-molecule_fig4_251220939)

- and many more

Extra conditions on graphs:

- **Simple:** No loops at nodes , and at most one edge between every two nodes

- **Tree:** No path starting and ending at the same node (cycle)

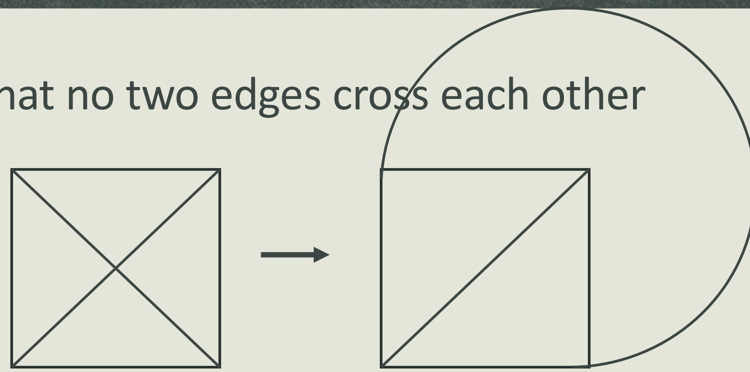
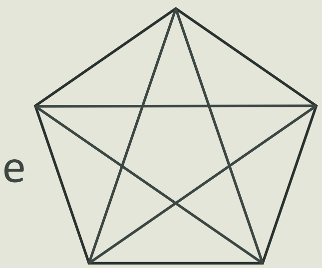
So at most one path between every two nodes

Trees

- **Lemma.** Every connected tree with N vertices has N-1 edges
- Prove it by induction on N.

Extra conditions on graphs:

- **Planar:** Can be drawn so that no two edges cross each other

- This can be drawn planar  but not this one 

- **Bipartite:** Nodes can be two groups so that there is no edges within each group

Characterization of bipartite graphs

- **Lemma.** A graph is bipartite if and only if it does not contain an odd **cycle**.

- Proof. Start from an arbitrary node and put it in group 1. Put any node connected to that in group 2, put any node connected to nodes in group 2 in group 1 again.

At some point you will revisit a node that is already labeled, but that wont cause a problem whenever there is no odd cycle. If all cycles have even length, the label you started with will be the same as the label you end with.

Planar graphs

- **Lemma (Euler's Formula):** For a planar graph with v Nodes, e Edges and f Faces we have

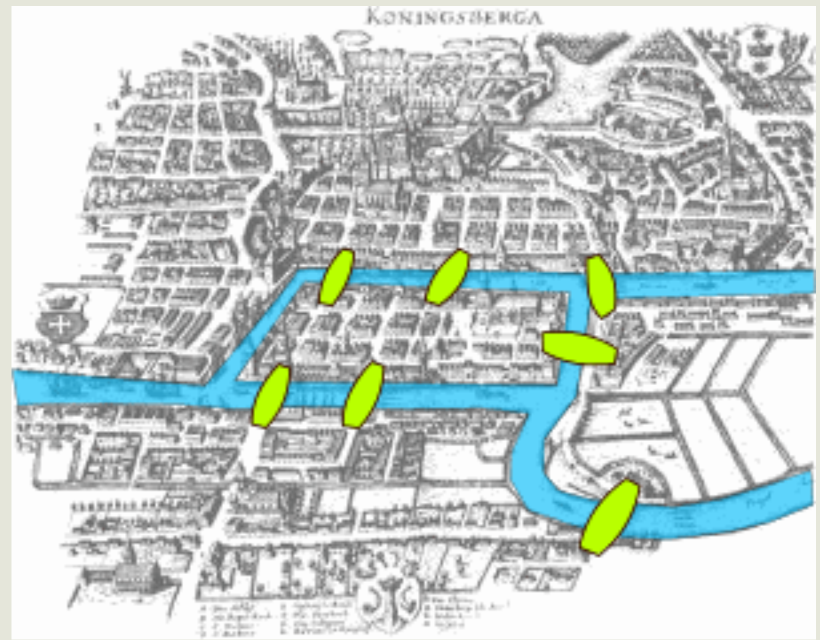
$$v - e + f = 2$$

<u>Faces</u>: different regions of the plane in the complement of the graph

Proof: prove it by induction on the number of vertices.

An ancient problem

- **Konigsberg bridges problem (**This is a historical problem in mathematics. Its negative resolution by Euler in 1736 laid the foundations of modern graph theory)
- **Question**: The city of **Konigsberg** in Prussia was set on both sides of a river, and included two large islands which were connected to each other, or to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.



* image: Courtesy of Wikipedia

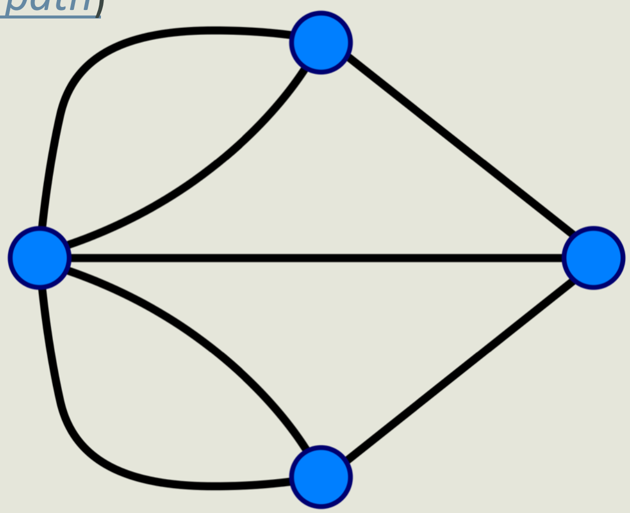
Konigsberg islands and bridges as a graph

- The problem is whether the following graph admits a path that covers each edge once and only once (Such a path is called an *Eulerian path*)
- Answer: No
- Proof:

degree of a node: number of edges connected to it.

Lemma: If a graph G admits an Eulerian path then it must have

NO odd degree node or TWO odd degree nodes



Comments

- Wikipedia:

Eulerian paths are used in bioinformatics to reconstruct the DNA sequence from its fragments.

They are also used in CMOS circuit design to find an optimal logic gate ordering.

Dual of planar graphs and a relation

- **Lemma**. A connected planar graph is Eulerian if and only if its dual is bipartite

A problem to practice

- **Q.** There are 25 telephones in Geeksland. Is it possible to connect them with wires so that each telephone is connected with exactly 7 others.

- Source: https://www.geeksforgeeks.org/graph-theory-practice-questions/

Links for reading

- Important Problems in Graph Theory: https://towardsdatascience.com/common-graph-theory-problems-ca990c6865f1
- Random Questions to think about:

http://www.geometer.org/mathcircles/graphprobs.pdf

- A free book: https://amsi.org.au/ESA_Senior_Years/PDF/PDFvcaa/graphtheory7a.pdf